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On a complete orthonormal system of special functions

E.I. Moiseev

Department of Computational, Mathematics and Cybernetics, Moscow State University, Moscow, Russian Federation

A.P. Prudnikov

Computer Center, Russian Academy of Sciences, Moscow, Russian Federation

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Abstract

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New complete orthonormal (CON) systems of special functions in the space of functions with summable squares are introduced.

Keywords: Complete orthonormal systems; special functions

1. On the Wiener orthonormal system

We first consider the Laplace transform of $t^{\alpha/2}$ times the Laguerre polynomials defined by the equation

$$\begin{aligned} l_n^\alpha(p) &= \int_0^\infty \mathrm{e}^{-pt} t^{\alpha/2} L_n^\alpha(t)\,\mathrm{d}t \\ &= \frac{\Gamma(1+\frac{1}{2}\alpha)\Gamma(\alpha+n+1)}{n!\,\Gamma(1+\alpha)p^{1+\alpha/2}}\,{}_2F_1\left(-n,\,1+\tfrac{1}{2}\alpha;\,\alpha+1;\,\frac{1}{p}\right). \end{aligned} \tag{1}$$

Taking into account the familiar Parseval's equation for the Laplace transform, we get

$$\frac{\Gamma(\alpha+n+1)}{n!}\delta_{mn} = \int_0^\infty L_n^\alpha(x)L_m^\alpha(x)x^\alpha \mathrm{e}^{-x}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^\infty l_n^\alpha(\tfrac{1}{2}+\mathrm{i}t)\overline{l_m^\alpha(\tfrac{1}{2}+\mathrm{i}t)}\,\mathrm{d}t,$$

Correspondence to: Dr. A.P. Prudnikov, Computer Center, Russian Academy of Sciences, Moscow, 117333, Russian Federation.

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where δ_{mn} is the Kronecker delta function and the bar denotes the complex conjugate. We therefore consider a system of orthonormalized functions

$$\rho_n^\alpha(\tau, a) = \left(\frac{n!}{4\pi a\Gamma(\alpha+n+1)}\right)^{1/2} l_n^\alpha\left(\frac{1}{2}+\frac{\mathrm{i}\tau}{2a}\right)$$
$$= A_n^\alpha {}_2F_1\left(-n, 1+\tfrac{1}{2}\alpha; \alpha+1; \left(\frac{1}{2}+\frac{\mathrm{i}\tau}{2a}\right)^{-1}\right)\left(\frac{1}{2}+\frac{\mathrm{i}\tau}{2a}\right)^{-1-\alpha/2}, \tag{2}$$

where

$$A_n^\alpha = \frac{1}{2}\frac{\Gamma(\frac{1}{2}\alpha+1)}{\Gamma(\alpha+1)}\left(\frac{F(\alpha+n+1)}{n!\pi a}\right)^{1/2}.$$

Higgins [4, p.64] has given a similar system of functions and he has proved that this system is complete in the subspace F^+, where F^+ is the space of the image of the Fourier transform of functions, which equals zero on the negative part of the real line.

It follows from (2), by making use of [1, p.101, (4)], that when $\alpha=0$ and $\alpha=1$, this system reduces to a system of functions

$$\rho_n(\tau) = \rho_n^0(\tau, 1) = \tfrac{1}{2}(\pi a)^{-1/2} {}_2F_1\left(-n, 1; 1; \left(\frac{1}{2}+\frac{\mathrm{i}\tau}{2a}\right)^{-1}\right)\left(\frac{1}{2}+\frac{\mathrm{i}\tau}{2a}\right)^{-1}\Bigg|_{a=1}$$
$$= \tfrac{1}{2}(\pi a)^{-1/2}\frac{\left(\mathrm{i}\tau/(2a)-\frac{1}{2}\right)^n}{\left(\mathrm{i}\tau/(2a)+\frac{1}{2}\right)^{n+1}}\Bigg|_{a=1} = \pi^{-1/2}\frac{(\mathrm{i}\tau-1)^n}{(\mathrm{i}\tau+1)^{n+1}}, \tag{3}$$

which was first introduced by Wiener [5, p.35]. Higgins [4, pp. 59–64], Christov [2, p.1339] and Glaeske [3, p.28] have briefly discussed the completeness and orthonormality of system (3). It follows from the familiar properties of the hypergeometric functions that $\rho_n^\alpha(\tau, a)$ is a rational function and can also be written in the form of the finite sum

$$\rho_n^\alpha(\tau, a) = A_n^\alpha \sum_{k=0}^{n} \frac{(-n)_k(1+\frac{1}{2}\alpha)_k}{(\alpha+1)_k\, k!\left(\frac{1}{2}+\mathrm{i}\tau/(2a)\right)^{k+1+\alpha/2}}$$
$$= A_n^\alpha \sum_{k=0}^{n} \frac{(-n)_k(1+\frac{1}{2}\alpha)_k \exp\left[-\mathrm{i}(k+1+\frac{1}{2}\alpha)\arctan(\tau/a)\right]}{(\alpha+1)_k\, k!\left(\frac{1}{4}+\tau^2/(4a^2)\right)^{(k+1+\alpha/2)/2}}. \tag{4}$$

We shall now write $C_n^\alpha(\tau, a)$ and $S_n^\alpha(\tau, a)$ for the following real functions:

$$C_n^\alpha(\tau, a) + \mathrm{i}S_n^\alpha(\tau, a) = 2^{1/2}\rho_n^\alpha(\tau, a). \tag{5}$$

It follows from equalities (4) and (5) that

$$S_n^\alpha(\tau, a) = -2^{1/2}A_n^\alpha \sum_{k=0}^{n} \frac{(-n)_k(1+\frac{1}{2}\alpha)_k \sin\left[(k+1+\frac{1}{2}\alpha)\arctan(\tau/a)\right]}{(\alpha+1)_k\, k!\left(\frac{1}{4}+\tau^2/(4a^2)\right)^{(k+1+\alpha/2)/2}},$$

$$C_n^\alpha(\tau, a) = 2^{1/2}A_n^\alpha \sum_{k=0}^{n} \frac{(-n)_k(1+\frac{1}{2}\alpha)_k \cos\left[(k+1+\frac{1}{2}\alpha)\arctan(\tau/a)\right]}{(\alpha+1)_k\, k!\left(\frac{1}{4}+\tau^2/(4a^2)\right)^{(k+1+\alpha/2)/2}}.$$

We obtain from relations (1), (2), (5),

$$C_n^\alpha(\tau, a) = \left(\frac{n!}{2\pi a\Gamma(\alpha+n+1)}\right)^{1/2} \int_0^\infty t^{\alpha/2} L_n^\alpha(t) \mathrm{e}^{-t/2} \cos\frac{\tau t}{2a}\, \mathrm{d}t,$$

$$S_n^\alpha(\tau, a) = -\left(\frac{n!}{2\pi a\Gamma(\alpha+n+1)}\right)^{1/2} \int_0^\infty t^{\alpha/2} L_n^\alpha(t) \mathrm{e}^{-t/2} \sin\frac{\tau t}{2a}\, \mathrm{d}t.$$

Let the set of functions with summable squares in the interval (a, b) be called $L^2(a, b)$, as usual. It follows from familiar properties of the Fourier sine and cosine transforms and from the fact that the $\{n!/\Gamma(\alpha+n+1)\}^{1/2}\mathrm{e}^{-t/2}t^{\alpha/2}$ times Laguerre polynomials constitute a complete orthonormal (CON) system in $L^2(0, \infty)$ that the systems $\sqrt{2}\,C_n^\alpha(\tau, a)$ and $\sqrt{2}\,S_n^\alpha(\tau, a)$ for $n = 0, 1, 2, \ldots$ are also CON systems in $L^2(0, \infty)$. This can be verified by using the Parseval's relation and the theorem of Higgins [4, p.15]. That is,

$$\delta_{mn} = 2\int_0^\infty C_n^\alpha(\tau, a) C_m^\alpha(\tau, a)\, \mathrm{d}\tau = 2\int_0^\infty S_n^\alpha(\tau, a) S_m^\alpha(\tau, a)\, \mathrm{d}\tau.$$

We at once get that the system of functions

$$\rho_n^\alpha(\tau, a) = \frac{\Gamma(\frac{1}{2}\alpha+1)}{2\Gamma(\alpha+1)} \left(\frac{\Gamma(\alpha+n+1)}{\pi a n!}\right)^{1/2} \times {}_2F_1\left(-n, 1+\tfrac{1}{2}\alpha; \alpha+1; \left(\frac{1}{2}+\frac{\mathrm{i}\tau}{2a}\right)^{-1}\right)\left(\frac{1}{2}+\frac{\mathrm{i}\tau}{2a}\right)^{-1-\alpha/2},$$

$$n = 0, 1, 2, \ldots,$$

$$\rho_{-n}^\alpha(\tau, a) = \overline{\rho_{n-1}^\alpha(\tau, a)}, \quad n = 1, 2, \ldots,$$

is a CON system in $L^2(-\infty, \infty)$. We shall state the main results in the following theorems.

Theorem 1. *The set* $\{2^{1/2}S_n^\alpha(\tau, a)$: $n = 0, 1, 2, \ldots,\ \tau \in (0, \infty),\ a > 0,\ \alpha > 0\}$ *forms a CON sequence in* $L^2(0, \infty)$.

Theorem 2. *The set* $\{2^{1/2}C_n^\alpha(\tau, a)$: $n = 0, 1, 2, \ldots,\ \tau \in (0, \infty),\ a > 0,\ \alpha > 0\}$ *forms a CON sequence in* $L^2(0, \infty)$.

Theorem 3. *The set* $\{\rho_n^\alpha(\tau, a)$: $n = 0, \pm 1, \pm 2, \ldots,\ \tau \in (-\infty, \infty),\ a > 0,\ \alpha > 0\}$ *forms a CON sequence in* $L^2(-\infty, \infty)$.

2. Differential equation and generating function

A differential equation for the functions $\rho_n^\alpha(\tau, a)$ may be obtained with the aid of the Laplace transform of the differential equation for the Laguerre polynomials. This gives us

$$\left[(a^2+\tau^2)\rho_n^\alpha\right]''_{\tau\tau} + \left[(\mathrm{i}a(\alpha+1+2n)-\tau)\rho_n^\alpha\right]'_\tau - \tfrac{1}{4}\alpha^2\rho_n^\alpha = 0.$$

On applying the Laplace transform on the recurrence formula of the Laguerre polynomials, the recurrence formula of the $\rho_n^\alpha(\tau, a)$ can be written in the form

$$\sqrt{(\alpha+n+1)(n+1)}\,\rho_{n+1}^\alpha-(2n+\alpha+1)\rho_n^\alpha+\sqrt{n(n+\alpha)}\,\rho_{n-1}^\alpha+2a\mathrm{i}\frac{\mathrm{d}\rho_n^\alpha}{\mathrm{d}\tau}=0,$$

$$\mathrm{i}\left[(a+\mathrm{i}\tau)\rho_n^\alpha\right]_\tau'=\left(n+\tfrac{1}{2}\alpha\right)\rho_n^\alpha-\sqrt{n(n+\alpha)}\,\rho_{n-1}^\alpha.$$

The generating function of this case reads as follows:

$$(1-z)^{-1-\alpha}\left(\frac{1}{2}+\frac{\mathrm{i}\tau}{2a}-\frac{z}{z-1}\right)^{-1-\alpha/2}$$

$$=\frac{2(\pi a)^{1/2}}{\Gamma\left(1+\frac{1}{2}\alpha\right)}\sum_{n=0}^{\infty}\left(\frac{\Gamma(\alpha+n+1)}{\Gamma(n+1)}\right)^{1/2}\rho_n^\alpha(\tau, a)z^n.$$

We have also seen from (2) by using formulae [1, p.108, (2)] and [1, p.102, (17)] that $\rho_n^\alpha(\tau, a)$ is expressible in the form

$$\rho_n^\alpha(\tau, a)=\frac{\Gamma\left(1+\frac{1}{2}\alpha\right)(1-p)^{n+\alpha/2}}{2(\pi a n!\,\Gamma(\alpha+n+1))^{1/2}}\frac{\mathrm{d}^n}{\mathrm{d}p^n}\left(\frac{1}{p^{\alpha/2+1}(1-p)^{\alpha/2}}\right),$$

where $p=\frac{1}{2}+\mathrm{i}\tau/(2a)$. The basic properties of the function $\rho_n^\alpha(\tau, a)$ at $\alpha=0$ and $a=1$ are given in [3]. As applications of these properties, it should be noted that

$$\rho_n(\tau)=\rho_n^0(\tau, 1)=(-1)^n\left[\pi(1+\tau^2)\right]^{-1/2}\exp[-\mathrm{i}(2n+1)\arctan\tau],$$
$$n=0,\ \pm1,\ \pm2,\ldots,\quad \tau\in(-\infty, \infty). \tag{6}$$

With the help of (6), the orthonormal system $\rho_n(\tau)$ on $(-\infty, \infty)$ can be verified by direct integration. In addition, with the change of variable $t=\arctan\tau$, a problem of pointwise convergence of Fourier series can be at once reduced to the convergence of trigonometrical series.

From (1) and (6) with the help of the inverse Laplace transform, the next integral representation of Laguerre polynomials follows:

$$L_n(t)=\frac{1}{2\pi\mathrm{i}}\int_{1/2-\mathrm{i}\infty}^{1/2+\mathrm{i}\infty}\mathrm{e}^{pt}l_n^0(p)\,\mathrm{d}p=\frac{1}{\mathrm{i}}\left(\frac{a}{\pi}\right)^{1/2}\int_{1/2-\mathrm{i}\infty}^{1/2+\mathrm{i}\infty}\mathrm{e}^{pt}\rho_n(p)\,\mathrm{d}p$$

$$=\frac{(-1)^n a\mathrm{e}^{t/2}}{\pi}\int_{-\infty}^{\infty}(a^2+\tau^2)^{-1/2}\exp\left[\frac{\mathrm{i}\tau t}{2a}-\mathrm{i}(2n+1)\arctan\frac{\tau}{a}\right]\mathrm{d}t$$

(here $p=\frac{1}{2}+\mathrm{i}\tau/(2a)$), that is,

$$\mathrm{e}^{-t/2}L_n(t)=\frac{(-1)^n a}{\pi}\int_{-\infty}^{\infty}(a^2+\tau^2)^{-1/2}\exp\left[\frac{\mathrm{i}\tau t}{2a}-\mathrm{i}(2n+1)\arctan\frac{\tau}{a}\right]\mathrm{d}\tau.$$

3. Another system of orthonormalized functions

Similar to (6), we can rewrite the orthogonal systems of functions considered in [4, p.64] in the form

$$\frac{1}{2(a\pi)^{1/2}} \frac{\left(\frac{1}{2} - \mathrm{i}\tau/(2a)\right)^{n+\alpha}}{\left(\frac{1}{2} + \mathrm{i}\tau/(2a)\right)^{n+\alpha+1}}$$

$$= \left[(\pi a)\left(1 + \left(\frac{\tau}{a}\right)^2\right)\right]^{-1/2} \exp\left[[-1 - 2(n+\alpha)]\mathrm{i} \arctan \frac{\tau}{a}\right].$$

On putting $a = \frac{1}{2}$, we find the inverse Fourier transform of these functions. Let $x > 0$ and $\alpha < -n$; then by deforming the contour of integration so that it coincides with the negative part of the real line and rotating about the origin, we get

$$I_n(x, \alpha) = \frac{1}{2\pi \mathrm{i}} \int_{1/2-\mathrm{i}\infty}^{1/2+\mathrm{i}\infty} \frac{\mathrm{e}^{px}(1-p)^{n+\alpha}}{p^{n+\alpha+1}} \, \mathrm{d}p$$

$$= \frac{\sin \pi(n+\alpha+1)}{\pi} \int_0^\infty \frac{\mathrm{e}^{-tx}(1+t)^{n+\alpha}}{t^{n+1+\alpha}} \, \mathrm{d}t = \frac{\psi(-n-\alpha; 1; x)}{\Gamma(n+\alpha+1)},$$

where $\psi(a; b; x)$ is the hypergeometric function of Tricomi [1, p.257]. By virtue of analytic continuation, it can be seen that the result remains valid for $\alpha \geqslant -n$.

Now let $x < 0$. In this case the contour of integration is deformed to the positive part of the real line and with a rotation about the origin we get

$$I_n(x, \alpha) = -\frac{\sin \pi(n+\alpha)}{\pi} \mathrm{e}^x \Gamma(n+\alpha+1) \psi(n+\alpha+1; 1; -x).$$

We therefore consider the system of orthonormalized functions

$$\mathscr{L}_n(x, \alpha) = \begin{cases} \dfrac{\mathrm{e}^{-x/2}}{\Gamma(n+\alpha+1)} \psi(-n-\alpha; 1; x), & x > 0, \\ -\dfrac{\mathrm{e}^{x/2} \sin \pi(n+\alpha)\, \Gamma(n+\alpha+1)}{\pi} \psi(n+\alpha+1; 1; -x), & x < 0. \end{cases}$$

If $\alpha = 0$, $n \geqslant 0$, then $\mathscr{L}_n(x, 0) = (-1)^n \mathrm{e}^{-x/2} L_n(x)$ for $x > 0$ and $\mathscr{L}_n(x, 0) = 0$ for $x < 0$. We have the following theorem.

Theorem 4. *The set* $\{\mathscr{L}_n(x, \alpha)$: $n = 0, \pm 1, \pm 2, \ldots,$ α *any real number*$\}$ *forms a CON sequence in* $L^2(-\infty, \infty)$.

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